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EXIT PROBABILITIES AND OPTIMAL STOCHASTIC CONTROL

by

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# EXIT PROBABILITIES AND OPTIMAL STOCHASTIC CONTROL

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Abstract: This paper is concerned with Markov diffusion processes which obey stochastic differential equations depending on a small parameter  $\epsilon$ . The parameter enters as a coefficient in the noise term of the stochastic differential equation. The Ventcel-Freidlin estimates give asymptotic formulas (as  $\epsilon \rightarrow 0$ ) for such quantities as the probability of exit from a region  $D$  through a given portion  $N$  of the boundary  $\partial D$ , the mean exit time, and the probability of exit by a given time  $T$ . A new method to obtain such estimates is given, using ideas from stochastic control theory.

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## EXIT PROBABILITIES AND OPTIMAL STOCHASTIC CONTROL

Wendell H. Fleming

### Accompanying Statement

This paper is concerned with Markov diffusion processes which obey stochastic differential equations depending on a small parameter  $\varepsilon$ . The parameter enters as a coefficient in the noise term of the stochastic differential equation. The Ventcel-Freidlin estimates give asymptotic formulas (as  $\varepsilon \rightarrow 0$ ) for such quantities as the probability of exit from a region  $D$  through a given portion  $N$  of the boundary  $\partial D$ , the mean exit time, and the probability of exit by a given time  $T$ . A new method to obtain such estimates is given, using ideas from stochastic control theory.

The method involves a logarithmic transformation on positive solutions  $g^\varepsilon(x)$  of the elliptic partial differential equation  $\mathcal{L}^\varepsilon g^\varepsilon = 0$ , where  $\mathcal{L}^\varepsilon$  is the generator of the diffusion process. In fact,  $J^\varepsilon = -\varepsilon \log g^\varepsilon$  satisfies a nonlinear elliptic partial differential equation, which is just the dynamic programming equation associated with the stochastic control problem. The boundary data  $\phi(x)$  for  $J^\varepsilon(x)$  are chosen according to a "penalty function" method; for example, if it is desired that exit occur through  $N \subset \partial D$ , then we take  $\phi(x) = 0$  on  $N$  and  $\phi(x)$  large on compact subsets of the complement of  $N$ . When  $\varepsilon = 0$  the dynamic programming equation reduces to the Hamilton-Jacobi equation

for the calculus of variations problem which enters in the Ventcel-Freidlin estimates. Thus, the stochastic control method provides not only a different way to prove such estimates, but also a different intuition to explain why one should expect them to be true.

## EXIT PROBABILITIES AND OPTIMAL STOCHASTIC CONTROL

Wendell H. Fleming

1. Introduction. Consider a Markov diffusion process  $\xi^\varepsilon$  on  $n$ -dimensional  $R^n$  which obeys the stochastic differential equations

$$(1.1) \quad d\xi^\varepsilon = b[\xi^\varepsilon(t)]dt + \sqrt{\varepsilon} \sigma[\xi^\varepsilon(t)]dw, \quad t \geq 0,$$

with initial data  $\xi^\varepsilon(0) = x$ . Here  $w$  is an  $n$ -dimensional brownian motion and  $\varepsilon$  a positive parameter.

Let  $\tau_D^\varepsilon$  denote the exit time of  $\xi^\varepsilon(t)$  from an open bounded set  $D \subset R^n$ . The Ventcel-Freidlin estimates give asymptotic formulas (as  $\varepsilon \rightarrow 0$ ) for such quantities as the probability  $P_x(\xi^\varepsilon(\tau_D^\varepsilon) \in N)$ , where  $N \subset \partial D$ , the mean exit time  $E_x \tau_D^\varepsilon$ , and the probability  $P_x(\tau_D^\varepsilon \leq T)$  of exit by a fixed time  $T$ . See [16], [17], [7, II, Chap. 14]. In this paper we give a new method to obtain such estimates, using ideas from optimal stochastic control theory. However, the paper is self-contained as far as knowledge of stochastic control is concerned.

There are two kinds of Ventcel-Freidlin estimates. The first kind give lower estimates for the probability that  $\xi^\varepsilon(t)$  remains in a given open set of curves for  $0 \leq t \leq T$ ; see [7, II, p. 332]. These lower estimates follow rather easily from the Girsanov transformation formula. At the end of the paper we outline a somewhat different derivation of a lower estimate.

The second kind of Ventcel-Freidlin results give upper estimates for the probability that  $\xi^\varepsilon(t)$  remains in a given closed set of curves [7, II, p. 334-345]. The proofs are more technical. Our main results provide estimates of the second kind, which suffice to study the problem of exit.

We begin in §2 by making a logarithmic transformation on positive solutions  $g^\varepsilon(x)$  of the elliptic partial differential equation  $\mathcal{L}^\varepsilon g^\varepsilon = 0$  in some region  $\Delta$ , where  $\mathcal{L}^\varepsilon$  is the generator of the diffusion process  $\xi^\varepsilon$ . In fact, let  $J^\varepsilon = -\varepsilon \log g^\varepsilon$ . Then  $J^\varepsilon(x)$  satisfies the nonlinear partial differential equation (2.5), with given boundary data  $J^\varepsilon = \phi$  on  $\partial D$ . Theorem 2.1 states that  $J^\varepsilon(x)$  is the minimum for a certain stochastic control problem, in which the drift coefficient  $b$  in (1.1) is replaced by a control process  $v(t)$ . A similar logarithmic transformation was used by E. Hopf [10] to solve Burger's equation. Recently, (and independently of our work) Holland [8], [9] used a logarithmic transformation of solutions to second order linear elliptic equations. He obtained a stochastic control representation of the dominant eigenvalue for Schrödinger's equation, and another proof of the Donsker-Varadhan formula for the dominant eigenvalue in case of natural boundary conditions.

The Ventcel-Freidlin estimates involve minimizing the following functional  $I(\phi, \theta)$  for various choices of  $R^n$ -valued functions  $\phi(t)$  and  $\theta \geq 0$ . Let

$$(1.2) \quad I(\phi, \theta) = \int_0^\theta L[\phi(t), \dot{\phi}(t)] dt,$$

where for  $x, v \in R^n$

$$(1.3) \quad L(x, v) = (b(x) - v)' a(x)^{-1} (b(x) - v),$$

$$a(x) = \sigma(x) \sigma'(x).$$

In particular, let  $\Delta$  be open, bounded, with  $x = \phi(0)$  in  $\Delta$ ; and let  $\theta$  be the exit time of  $\phi(t)$  from  $\Delta$ . For  $N \subset \partial\Delta$  let  $I_\Delta(x, N)$  denote the infimum of  $I(\phi, \theta)$  subject to the additional condition  $\phi(\theta) \in N$ . Theorem 5.1 implies that

$$-I_\Delta(x, N) = \lim_{\varepsilon \rightarrow 0} \varepsilon \log P_x(\xi^\varepsilon(\tau_\Delta^\varepsilon) \in N),$$

under some rather stringent hypotheses on  $b$  and  $\Delta$ . As in known [16], [17], [7, II, pp. 386-387] one can then apply Theorem 5.1 to get results on the exit place and exit time of  $\xi^\varepsilon(t)$  from a region  $D$ , under certain assumptions on the behavior as  $t \rightarrow \infty$  of solutions of the unperturbed system  $\dot{\xi}^0 = b[\xi^0(t)]$ . We include one such result in §6.

In §3 we consider instead of (1.2) the corresponding functional  $J(\phi, \theta)$  obtained by adding a term  $\Phi[\phi(\theta)]$  on the right side of (1.2). If  $\Phi(y) = 0$  for  $y \in N$  and  $\Phi \geq 0$ , we may regard  $\Phi[\phi(\theta)]$  as a penalty imposed if  $\phi(\theta) \notin N$  at the exit time  $\theta$ . Later (in §5) we take a sequence  $\Phi_M$  of penalty functions such that  $\Phi_M(y) \rightarrow +\infty$  as  $M \rightarrow \infty$  for  $y \notin \bar{N}$ .

In §7 we consider the nonautonomous form of (1.1) on a finite time interval  $s \leq t \leq T$ . There is a considerably simpler proof of the main result (Theorem 7.1); and no assumptions on the large-time behavior of the unperturbed system are needed.

2. The logarithmic transformation. With the stochastic differential equation (1.1) is associated the differential generator  $\mathcal{L}^\varepsilon$ , namely,

$$\mathcal{L}^\varepsilon g = \frac{\varepsilon}{2} \sum_{j,k=1}^n a_{jk}(x) g_{x_j x_k} + g_x \cdot b(x).$$

Here  $a = (a_{jk}) = \sigma \sigma'$ , and  $g_x$  is the gradient. We assume:

(A<sub>1</sub>)  $b(x), \sigma(x)$ , and the inverse  $\sigma^{-1}(x)$  are bounded, Lipschitz functions on  $R^n$ .

Since  $\sigma^{-1}$  is bounded,  $\mathcal{L}^\varepsilon$  is a uniformly elliptic operator for each  $\varepsilon > 0$ .

Let  $\Delta \subset R^n$  be an open, bounded set, with  $C^2$  boundary  $\partial\Delta$  (i.e.,  $\partial\Delta$  is a manifold of class  $C^2$ .) Let  $\phi$  be of class  $C^2$ , and  $\phi \geq 0$ . Consider the boundary value problem

$$(2.1) \quad \mathcal{L}^\varepsilon g^\varepsilon = 0 \quad \text{in } \Delta$$

$$(2.2) \quad g^\varepsilon(x) = \exp\left[-\frac{\phi(x)}{\varepsilon}\right] \quad \text{on } \partial\Delta.$$

There is a unique solution  $g^\varepsilon$ , of class  $C^2(\Delta) \cap C^1(\bar{\Delta})$ ,  $\bar{\Delta} = \Delta \cup \partial\Delta$ . See [11, Chap. III, §12, 15]. Moreover,

$$(2.3) \quad g^\varepsilon(x) = E_x \exp\left\{-\frac{\phi[\xi^\varepsilon(\tau_\Delta^\varepsilon)]}{\varepsilon}\right\},$$

where  $\tau_\Delta^\varepsilon$  is the exit time from  $\Delta$  of  $\xi^\varepsilon(t)$ .

We make the following logarithmic transformation. Let

$$(2.4) \quad J^\varepsilon(x) = -\varepsilon \log g^\varepsilon(x).$$

By elementary calculus,  $J^\varepsilon$  satisfies in  $\Delta$  the nonlinear elliptic equation

$$(2.5) \quad 0 = \frac{\varepsilon}{2} \sum_{j,k=1}^n a_{jk}(x) J_{x_j x_k}^\varepsilon + H(x, J_x^\varepsilon),$$

where for each  $x$  and row vector  $p$

$$(2.6) \quad H(x, p) = -\frac{1}{2} p a(x) p' + p \cdot b(x).$$

Then  $H(x, \cdot)$  is dual to  $L(x, \cdot)$ , where  $L$  was defined by (1.3), in the sense of duality for concave and convex functions. In particular, from (1.3) and (2.6)

$$(2.7) \quad H(x, p) = \min_v [L(x, v) + p \cdot v].$$

Equation (2.5) is the dynamic programming equation for the following optimal stochastic control problem. Let  $\eta(t)$  denote the state of a system being controlled for  $t \geq 0$ , and  $v(t)$  the control used at time  $t$  ( $\eta(t) \in \mathbb{R}^n$ ,  $v(t) \in \mathbb{R}^n$  for each  $t$ ). These processes are defined on some probability space  $(\Omega, \mathcal{F}, P)$  and are nonanticipative with respect to some increasing family  $\{\mathcal{F}_t\}$  of  $\sigma$ -algebras,  $\mathcal{F}_t \subset \mathcal{F}$ . We assume that the control  $v(t)$  is bounded. The state  $\eta$  satisfies the stochastic differential equation

$$(2.8) \quad d\eta = v(t)dt + \sqrt{\varepsilon} \sigma[\eta(t)]dw$$

with  $\eta(0) = x$ , where  $x \in \Delta$  and  $w$  is some brownian motion adapted to  $\{\mathcal{F}_t\}$ .

Let  $\theta$  denote the exit time of  $\eta(t)$  from  $\Delta$ , and let

$$(2.9) \quad J^\varepsilon(x, v) = E \left\{ \int_0^\theta L[\eta(t), v(t)] dt + \phi[\eta(\theta)] \right\}.$$

The stochastic control problem is to minimize  $J^\varepsilon(x, v)$  given the initial state  $x$ .

From the fact that  $J^\varepsilon(x)$  is a solution in class  $C^2(\Delta) \cap C^1(\bar{\Delta})$  of the dynamic programming equation (2.5), with  $J^\varepsilon = \phi$  on  $\partial\Delta$ , one can show that  $J^\varepsilon(x)$  is the minimum of  $J^\varepsilon(x, v)$  and can characterize an optimal control process  $v^\varepsilon(t)$ . This result is called a verification Theorem [6, p. 164], and is stated below as Theorem 2.1. For completeness we include the easy proof, as follows.

From (2.5) and (2.8)

$$(2.10) \quad \frac{\varepsilon}{2} \sum_{j,k=1}^n a_{jk} J_{x_j x_k}^\varepsilon + J_x^\varepsilon \cdot v(t) + L[\eta(t), v(t)] \geq 0,$$

where  $a_{jk}, J_{x_j x_k}^\varepsilon, J_x^\varepsilon$  are evaluated at  $\eta(t)$ . We apply the Itô stochastic differential rule to  $J^\varepsilon[\eta(t)]$  and take  $E \int_0^\theta \dots dt$ :

$$(2.11) \quad J^\varepsilon(x) \leq E \int_0^\theta L[\eta(t), v(t)] dt + E J^\varepsilon[\eta(\theta)].$$

Since  $J^\varepsilon = \phi$  on  $\partial\Delta$ ,

$$(2.12) \quad J^\varepsilon(x) \leq J^\varepsilon(x, v).$$

If  $v$  is a function from  $\Delta$  into  $R^n$ , we say that the control process  $v(t)$  is obtained from the feedback control law  $V$ , for initial data  $\eta(0) = x$ , if

$$(2.13) \quad v(t) = V[\eta(t)], \quad 0 \leq t < \theta.$$

For  $t \geq \theta$  we set  $v(t) = 0$ ; this does not affect  $J^\varepsilon(x; v)$ . We admit any feedback  $V$  which is bounded and locally Lipschitz on  $\Delta$ .

In particular, let

$$(2.14) \quad V^\varepsilon(x) = H_p(x, J_x^\varepsilon(x)) = b(x) - a(x)J_x^\varepsilon(x)'. \quad .$$

Let  $v^\varepsilon(t), \eta^\varepsilon(t)$  be the corresponding control process and solution of (2.8), for given initial data  $\eta^\varepsilon(0) = x$ . Equality holds in (2.10) when  $v = v^\varepsilon, \eta = \eta^\varepsilon$ . Then equality also holds in (2.11) when  $\theta = \theta^\varepsilon$ , with  $\theta^\varepsilon$  the exit time of  $\eta^\varepsilon$  from  $\Delta$ . Therefore,

$$(2.15) \quad J^\varepsilon(x) = J^\varepsilon(x; v^\varepsilon).$$

From (2.12) and (2.15) we have:

Theorem 2.1.  $J^\varepsilon(x)$  is the minimum of  $J^\varepsilon(x, v)$ , and  $V^\varepsilon$  is an optimal feedback control law.

3. A deterministic minimum problem. When  $\varepsilon = 0$  we consider the following minimum problem. Let  $C^n[0, \infty)$  denote the space of continuous,  $R^n$ -valued functions on  $[0, \infty)$ , and  $\mathcal{U}^1$  the space of

all  $\phi \in C^n[0, \infty)$  such that  $\phi$  is absolutely continuous and  $\int_0^T |\dot{\phi}(t)|^2 dt < \infty$  for each  $T > 0$ . For  $\phi \in \mathcal{A}^1$  and  $\theta \geq 0$  let

$$(3.1) \quad \mathcal{J}(\phi, \theta) = \int_0^\theta L[\phi(t), \dot{\phi}(t)] dt + \phi[\phi(\theta)].$$

Let  $\Delta$  be open bounded, with  $C^2$  boundary  $\partial\Delta$ ; and let  $\phi$  be of class  $C^2, \phi \geq 0$ . Given  $x \in \Delta$  let

$$(3.2) \quad J(x) = \inf_{\phi, \theta} \mathcal{J}(\phi, \theta),$$

where the infimum is taken among all  $\phi, \theta$  such that  $\phi(0) = x$ ,  $\phi(t) \in \bar{\Delta}$  for  $0 \leq t \leq \theta$ ,  $\phi(\theta) \in \partial\Delta$ . Under condition  $(B_1)$  below the minimum is attained in (3.2), but we shall not use this fact. Note that if we set  $\varepsilon = 0$ ,  $\eta(t) = \phi(t)$ ,  $v(t) = \dot{\phi}(t)$  in (2.8), then (2.9) reduces to (3.1). Moreover, when  $\varepsilon = 0$  the dynamic programming equation (2.5) reduces to  $0 = H(x, J_x)$ . This is the Hamilton-Jacobi equation associated with (3.2). One might expect therefore that  $J^\varepsilon(x)$  tends to  $J(x)$  as  $\varepsilon \rightarrow 0$ . We shall prove in §4 a partial result of this kind. See Lemma 4.2 and the Note, end of §4.

In §'s 3-5 we make the following assumption about the function  $b$  and the region  $\Delta$ .

$$(B_1) \quad \text{If } \phi \in \mathcal{A}^1 \text{ and } \phi(t) \in \bar{\Delta} \text{ for all } t \geq 0, \text{ then} \\ \int_0^\infty L[\phi(t), \dot{\phi}(t)] dt = +\infty.$$

Let  $\xi^0(t; x)$  denote the solution of

$$(3.3) \quad \frac{d\xi^0}{dt} = b[\xi^0(t; x)], \quad t \geq 0$$

with  $\xi^0(0; x) = x$ . Note that (3.3) is the unperturbed version of (1.1), with  $\varepsilon = 0$ . Let us show that  $(B_1)$  is implied by the following property  $(B_2)$ . For  $\alpha > 0$ , let  $\tilde{\Delta}_\alpha = \{y: \text{dist}(y, \Delta) > \alpha\}$ .

$(B_2)$  There exist  $T_1 > 0$ ,  $\alpha > 0$  with the following property: for every  $x \in \Delta$ , there exists  $t \in [0, T_1]$  such that  $\xi^0(t; x) \in \tilde{\Delta}_\alpha$ .

Lemma 3.1.  $(B_2)$  implies  $(B_1)$ .

Proof. Suppose that  $\phi \in \mathcal{A}^1$  and  $\phi(t) \in \bar{\Delta}$  for all  $t \geq 0$ . Let  $T_1, \alpha$  be as in  $(B_2)$ ; and for  $j = 0, 1, 2, \dots$  let

$$\xi_j^0(t) = \xi^0(t; \phi(jT_1)), \quad \phi_j(t) = \phi(t + jT_1).$$

Let  $M$  be a Lipschitz constant for  $b$ , and let  $|||_t$  denote the sup norm on  $[0, t]$ . Then, for  $0 \leq t \leq T_1$ ,

$$\begin{aligned} \phi_j(t) - \xi_j^0(t) &= \int_0^t [\dot{\phi}_j - b(\phi_j)] ds + \int_0^t [b(\phi_j) - b(\xi_j^0)] ds, \\ |||\phi_j - \xi_j^0|||_t &\leq t^{\frac{1}{2}} \left( \int_0^t |\dot{\phi}_j - b(\phi_j)|^2 ds \right)^{\frac{1}{2}} + M \int_0^t |||\phi_j - \xi_j^0|||_s ds. \end{aligned}$$

Moreover, by (1.3),  $|v - b(y)|^2 \leq CL(y, v)$  for some  $C$ . By Gronwall's inequality

$$|||\phi_j - \xi_j^0|||_{T_1} \leq C_1 \left( \int_0^{T_1} L(\phi_j, \dot{\phi}_j) dt \right)^{\frac{1}{2}},$$

with  $C_1 = (CT_1)^{\frac{1}{2}} \exp MT_1$ . However,  $\alpha \leq |||\phi_j - \xi_j^0|||_{T_1}$  by  $(B_2)$ ,

with  $x$  replaced by  $\phi(jT_1)$ , since  $\phi_j(t) \in \bar{\Delta}$  for  $0 \leq t \leq T_1$ .

Thus

$$\alpha^2 C_1^{-2} \leq \int_{jT_1}^{(j+1)T_1} L(\phi, \dot{\phi}) dt, \quad j = 0, 1, 2, \dots$$

Hence,  $\int_0^\infty L(\phi, \dot{\phi}) dt = +\infty$ , which proves Lemma 3.1.

Lemma 3.1 has the following simple interpretation. Condition  $(B_2)$  says that a particle following the flow (3.3) is swept out of  $\bar{\Delta}$  in bounded time. If  $L(\phi, \dot{\phi})$  is interpreted as a rate at which energy is expended to resist the flow, then infinite energy is required to remain in  $\Delta$  indefinitely.

4. A semicontinuity result. In this section we consider lower semicontinuity of  $J^\epsilon(x)$  as a function of  $\epsilon$  and  $x$ , at  $\epsilon = 0$ . For this purpose let us consider initial state  $x^\epsilon$  tending to a limit  $x^0$  as  $\epsilon \rightarrow 0$  ( $x^\epsilon \in \Delta$ ). As in §2, let  $v^\epsilon, \eta^\epsilon$  be the optimal control process and corresponding solution of (2.8) with  $\eta^\epsilon(0) = x^\epsilon$ . Recall that  $v^\epsilon(t)$  is obtained via (2.13) from the feedback control law  $V^\epsilon$ , for  $0 \leq t < \theta^\epsilon$ , and  $v^\epsilon(t) = 0$  for  $t \geq \theta^\epsilon$ .

Consider any sequence  $\epsilon_n$ , tending to 0 as  $n \rightarrow \infty$ , and let  $v^{\epsilon_n} = v_n$ ,  $x^{\epsilon_n} = x_n$ , etc. We then have

$$(4.1) \quad \eta_n = \phi_n + \sqrt{\epsilon_n} \zeta_n, \text{ where}$$

$$\phi_n(t) = x_n + \int_0^t v_n(s) ds, \quad \zeta_n(t) = \int_0^t \sigma[\eta_n(s)] dw.$$

We give  $C^n[0, \infty)$  the following metric, equivalent to uniform

convergence on each finite interval  $[0, T]$ :

$$d(\phi, \psi) = \sum_{j=1}^{\infty} 2^{-j} \frac{||\phi - \psi||_j}{1 + ||\phi - \psi||_j}.$$

For  $\Gamma \subset C^n[0, \infty)$  let  $\Gamma^T$  denote the set of restrictions  $\phi^T$  to  $[0, T]$  of functions  $\phi \in \Gamma$ . Then  $\Gamma$  is totally bounded in  $C^n[0, \infty)$  if and only if  $\Gamma^T$  is totally bounded in  $C^n[0, T]$  for each  $T$ . In the next lemma we verify the Prokhorov compactness condition for the sequences  $\phi_n, \zeta_n$ .

Lemma 4.1. Assume that the sequence  $J^{\varepsilon_n}(x_n)$  is bounded.  
Then for every  $\delta > 0$  there exist totally bounded sets  $\Gamma_{1\delta}, \Gamma_{2\delta}$   
such that  $P(\phi_n \in \Gamma_{1\delta}) > 1 - \delta, P(\zeta_n \in \Gamma_{2\delta}) > 1 - \delta, n = 1, 2, \dots$ .

Proof. Since  $v_n(t)$  is an optimal control process

$$(4.2) \quad J^{\varepsilon_n}(x_n) = E \left\{ \int_0^{\theta_n} L[\eta_n(t), v_n(t)] dt + \phi[\eta_n(\theta_n)] \right\}.$$

Since  $\phi \geq 0$ ,  $J^{\varepsilon_n}(x_n)$  is bounded, and  $L(y, v) \geq c|v - b(y)|^2$  for some  $c > 0$ ,  $E \int_0^{\theta_n} |v_n(t) - b[\eta_n(t)]|^2 dt$  is bounded. Since  $v_n(t) = 0$  for  $t \geq \theta_n$ , this implies

$$E \int_0^T |v_n(t)|^2 dt \leq C_T,$$

for some  $C_T$ . Let

$$\Gamma_{1\delta}^j = \{\phi: \phi(0) \in \Delta, |\phi(t) - \phi(s)|^2 \leq C_j 2^{j\delta-1}(t-s) \text{ for } 0 \leq s, t \leq j\}.$$

$$\Gamma_{1\delta} = \{\phi: \phi^j \in \Gamma_{1\delta}^j, j = 1, 2, \dots\}.$$

Then  $\Gamma_{1\delta}$  is totally bounded. By Cauchy-Schwartz,

$$\int_0^j |v_n(t)|^2 dt \leq C_j 2^{j\delta-1} \text{ implies } \phi_n^j \in \Gamma_{1\delta}^j. \text{ Hence}$$

$$P(\phi_n^j \notin \Gamma_{1\delta}^j) \leq 2^{-j\delta}, j = 1, 2, \dots,$$

$$P(\phi_n \notin \Gamma_{1\delta}) \leq \delta.$$

The existence of  $\Gamma_{2\delta}$  with the required property follows from the assumption that  $\sigma$  is a bounded function; see [13, Proposition 9]. This proves Lemma 4.1.

Let  $\gamma_n = \theta_n(1+\theta_n)^{-1}$ , and note that  $0 \leq \gamma_n < 1$ . Consider the triple  $(\phi_n, \zeta_n, \gamma_n)$ , regarded as a  $C^{2n}[0, \infty) \times [0, 1]$ -valued random variable. By Lemma 4.1 there is a subsequence, denoted again by  $(\phi_n, \zeta_n, \gamma_n)$ , for which the joint probability distribution measures converge weakly. By Skorokhod's theorem [14] there exist  $(\tilde{\phi}_n, \tilde{\zeta}_n, \tilde{\gamma}_n)$ , defined on the sample space  $\tilde{\Omega} = [0, 1]$  with the same distribution as  $(\phi_n, \zeta_n, \gamma_n)$ , such that with probability 1  $\tilde{\phi}_n, \tilde{\zeta}_n$  tend in d-metric to limits  $\tilde{\phi}, \tilde{\zeta}$  and  $\tilde{\gamma}_n$  tends to a random variable  $\tilde{\gamma}$  as  $n \rightarrow \infty$ .

Let  $\tilde{\eta}_n = \tilde{\phi}_n + \sqrt{\varepsilon_n} \tilde{\zeta}_n$ . Then  $d(\tilde{\eta}_n, \tilde{\phi}) \rightarrow 0$  with probability 1. The exit time  $\tilde{\theta}_n$  of  $\tilde{\eta}_n(t)$  from  $\Delta$  satisfies  $\tilde{\gamma}_n = \tilde{\theta}_n(1+\tilde{\theta}_n)^{-1}$ . Thus  $\tilde{\gamma} < 1$  if and only if  $\tilde{\theta}_n$  tends to a finite limit  $\tilde{\theta}$  as  $n \rightarrow \infty$ . We also have  $\tilde{\phi}(0) = x^0$ ,  $\tilde{\phi}(t) \in \bar{\Delta}$  for  $0 \leq t \leq \tilde{\theta}$ , and  $\tilde{\phi}(\tilde{\theta}) \in \partial\Delta$  if  $\tilde{\theta} < \infty$ .

For  $\eta, \phi \in C^n[0, \infty)$  and  $0 \leq \gamma \leq 1$ , define

$$h(\phi, \zeta, \gamma) = \int_0^{\theta} L[\eta(t), \dot{\phi}(t)] dt, \quad \gamma = \frac{\theta}{1+\theta},$$

if  $\phi \in \mathcal{A}^1$ ; otherwise let  $h(\eta, \phi, \gamma) = +\infty$ . Then  $h$  is lower semicontinuous on  $C^{2n}[0, \infty) \times [0, 1]$ . This is a well known result; it can be proved in the same way as [7, II, Lemma 1.2, p. 329]. The random variables  $h(\eta_n, \phi_n, \gamma_n)$  and  $h(\tilde{\eta}_n, \tilde{\phi}_n, \tilde{\gamma}_n)$  have the same distribution. Moreover, with probability 1

$$(4.3) \quad \liminf_{n \rightarrow \infty} \int_0^{\tilde{\theta}_n} L(\tilde{\eta}_n, \dot{\tilde{\phi}}_n) dt \geq \int_0^{\tilde{\theta}} L(\tilde{\phi}, \dot{\tilde{\phi}}) dt.$$

Lemma 4.2. Assume  $(B_1)$ . Then for any  $x^0 \in \Delta$ ,

$$\liminf_{\substack{\epsilon \rightarrow 0 \\ x \rightarrow x^0}} J^\epsilon(x) \geq J(x^0).$$

Proof. It suffices to show that

$$\liminf_{n \rightarrow \infty} J^{\epsilon_n}(x_n) \geq J(x^0)$$

for any sequences  $\epsilon_n, x_n$  tending to  $0, x^0$  with  $J^{\epsilon_n}(x_n)$  bounded. Using the notation above, we may suppose, moreover, that  $\tilde{\eta}_n, \tilde{\phi}_n$  tend to  $\tilde{\phi}$  in d-metric and  $\tilde{\theta}_n \rightarrow \tilde{\theta}$ , with probability 1. In (4.2) we replace  $\eta_n, \phi_n, \theta_n$  by  $\tilde{\eta}_n, \tilde{\phi}_n, \tilde{\theta}_n$  and recall that  $v_n = \dot{\phi}_n$ . By Fatou's lemma

$$(4.4) \quad \liminf_{n \rightarrow \infty} J^{\epsilon_n}(x_n) \geq E \liminf_{n \rightarrow \infty} \int_0^{\tilde{\theta}_n} L(\tilde{\eta}_n, \dot{\tilde{\phi}}_n) dt + \phi[\tilde{\eta}_n(\tilde{\theta}_n)].$$

Since  $J^{\epsilon_n}(x_n)$  is bounded and  $\phi \geq 0$ , (4.3) implies

$\int_0^{\tilde{\theta}} L(\tilde{\phi}, \dot{\tilde{\phi}}) dt < \infty$  with probability 1. Since  $\tilde{\phi}(t) \in \bar{\Delta}$  for  $0 \leq t < \tilde{\theta}$ , property  $(B_1)$  implies  $\tilde{\theta} < \infty$  with probability 1. Moreover,  $\tilde{\eta}_n(\tilde{\theta}_n) \rightarrow \tilde{\phi}(\tilde{\theta})$  with probability 1, and  $\tilde{\phi}(\tilde{\theta}) \in \partial\Delta$ . From (3.1), (4.4), and the fact that  $\phi$  is continuous on  $\partial\Delta$  we then have

$$\liminf_{n \rightarrow \infty} J^{\epsilon_n}(x_n) \geq E \mathcal{J}(\tilde{\phi}, \tilde{\theta}).$$

Since  $\tilde{\phi}(0) = x^0$ , we have  $\mathcal{J}(\tilde{\phi}, \tilde{\theta}) \geq J(x^0)$  with probability 1, and hence  $E \mathcal{J}(\tilde{\phi}, \tilde{\theta}) \geq J(x^0)$ . This proves Lemma 4.2.

Note. It can be shown that  $J^{\epsilon}(x) \rightarrow J(x)$  as  $\epsilon \rightarrow 0$ , although we do not use this fact. A proof that  $\limsup_{\epsilon \rightarrow 0} J^{\epsilon}(x) \leq J(x)$  can be given by the method at the end of §7. The essential idea is that given  $\phi$ , the deterministic control  $v(t) = \dot{\phi}(t)$  can be considered as a (suboptimal) control in the stochastic problem in §2. A similar comparison technique was used in [3, §5].

5. The exit problem (preliminary result). Let us return to the exit problem for solutions  $\xi^{\epsilon}(t)$  to (1.1). In the present section we consider a bounded open set  $\Delta$ , with  $C^2$  boundary  $\partial\Delta$ , for which the rather strong assumption  $(B_1)$  in §3 holds. We prove Theorem 5.1, which is then used for the main result in §6.

For  $N \subset \partial\Delta$ ,  $x \in \Delta$  let

$$(5.1) \quad q_{\Delta}^{\epsilon}(x, N) = P_x(\xi^{\epsilon}(\tau_{\Delta}^{\epsilon}) \in N),$$

$$(5.2) \quad I_{\Delta}^{\epsilon}(x, N) = -\epsilon \log q_{\Delta}^{\epsilon}(x, N),$$

with  $\tau_{\Delta}^{\varepsilon}$  the exit time from  $\Delta$  of  $\xi^{\varepsilon}(t)$ . Let

$$(5.3) \quad I_{\Delta}(x, N) = \inf_{\phi, \theta} \int_0^{\theta} L(\phi(t), \dot{\phi}(t)) dt,$$

where the infimum is taken among all  $\phi \in \mathcal{A}^1$  and  $\theta$  such that  $\phi(0) = x$ ,  $\phi(t) \in \bar{\Delta}$  for  $0 \leq t \leq \theta$ , and  $\phi(\theta) \in N$ . The infimum may also be taken in the set of  $(\phi, \theta)$  for which, in addition,  $\phi(t) \in \Delta$  for  $0 \leq t < \theta$  (and hence  $\theta$  is the exit time from  $\Delta$  for  $\phi(t)$ .) We also have

$$(5.4) \quad I_{\Delta}(x, N) = I_{\Delta}(x, \bar{N}).$$

For  $M = 1, 2, \dots$ , we introduce a "penalty function"  $\phi_M \geq 0$  such that  $\phi_M$  is class  $C^2$ ,  $\phi_M(x) \rightarrow +\infty$  as  $M \rightarrow \infty$  uniformly on any compact subset of  $R^n - \bar{N}$  and  $\phi_M(x) = 0$  on  $\bar{N}$ . The following lemma is easily proved, using lower semicontinuity of  $\int_0^{\theta} L(\phi, \dot{\phi}) dt$  with respect to  $\phi$  in the d-metric and with respect to  $\theta$ . Write  $J = J^M$  for the function in (3.2), when  $\Phi = \phi_M$ .

Lemma 5.1. Assume  $(B_1)$ . Then

$$\liminf_{\substack{M \rightarrow \infty \\ x \rightarrow x^0}} J^M(x) \geq I_{\Delta}(x^0, \bar{N}).$$

Let  $N^0$  denote the interior of  $N$ , relative to  $\partial\Delta$ .

Theorem 5.1. Assume  $(B_1)$  and that  $\bar{N} = \overline{N^0}$ . Then

$$\lim_{\varepsilon \rightarrow 0} I_{\Delta}^{\varepsilon}(x, N) = I_{\Delta}(x, N), \text{ uniformly for } x \text{ in any compact set } \Delta \subset \Delta.$$

Proof. It suffices to show that, uniformly for  $x \in \Lambda$ ,

$$(5.5) \quad \limsup_{\varepsilon \rightarrow 0} I_{\Delta}^{\varepsilon}(x, N) \leq I_{\Delta}(x, N^0)$$

$$(5.6) \quad \liminf_{\varepsilon \rightarrow 0} I_{\Delta}^{\varepsilon}(x, N) \geq I_{\Delta}(x, \bar{N}).$$

Note that  $I_{\Delta}(x, N^0) = I_{\Delta}(x, \bar{N})$  by (5.4), since  $\bar{N} = \overline{N^0}$ . Inequality (5.5) is an easy consequence of the first Ventcel-Freidlin estimate [7, p. 332]. We shall indicate another proof of (5.5) at the end of §7.

To prove (5.6) we introduce the penalty functions  $\phi_M$  above, and write  $g^{\varepsilon} = g^{\varepsilon M}$ ,  $J^{\varepsilon} = J^{\varepsilon M}$  when  $\phi = \phi_M$  (§2). Since  $\phi_M(y) = 0$  for  $y \in N$ , we have by (2.3) and (5.1),  $q_{\Delta}^{\varepsilon}(x, N) \leq g^{\varepsilon M}(x)$  for each  $M$ . By taking logarithms,  $J^{\varepsilon M}(x) \leq I_{\Delta}^{\varepsilon}(x, N)$ . By Lemmas 4.2 and 5.1, inequality (5.6) holds uniformly for  $x \in \Lambda$ . This proves Theorem 5.1.

6. The problem of exit (continued). Let  $D$  be a bounded open set, with  $C^2$  boundary  $\partial D$ . We illustrate the use of Theorem 5.1 by deriving an asymptotic result about the exit place of  $\xi^{\varepsilon}(t)$  (Theorem 6.1). With minor variations, the construction used to prove Theorem 6.1 is the same as in [17, p. 272] [7, pp. 386-387].

For  $x, y \in \bar{D}$  let

$$I_D(x, y) = \min_{\phi, \theta} \int_0^{\theta} L[\phi(t), \dot{\phi}(t)] dt$$

where the minimum is taken among  $\phi \in \mathcal{W}^1$ ,  $\theta \geq 0$  such that  $\phi(0) = x$ ,  $\phi(\theta) = y$ ,  $\phi(t) \in \bar{D}$  for  $0 \leq t \leq \theta$ . By (5.3), for

$$x \in D, N \subset \partial D$$

$$I_D(x, N) = \inf_{y \in N} I_D(x, y) = \min_{y \in \bar{N}} I_D(x, y).$$

We make the following assumptions (cf. [7, pp. 359-360]):

(B<sub>3</sub>)  $b(y) \cdot v(y) < 0$  for every  $y \in \partial D$ , where  $v(y)$  is the exterior normal.

(B<sub>4</sub>) There exists a compact set  $K \subset D$  such that:

(i)  $K$  contains the  $\omega$ -limit set of the solution  $\xi^0(t; x)$  of (3.3) for each  $x \in D - K$ .

(ii)  $I_D(x_1, x_2) = 0$  for every  $x_1, x_2 \in K$ .

(iii) Let  $K_\mu$  denote the  $\mu$ -neighborhood of  $K$ , and  $\Delta_\mu = D - \bar{K}_\mu$ . Then there exists  $c_\mu$  tending to 0 as  $\mu \rightarrow 0$  such that

$$I_{\Delta_\mu}(x, y) \leq I_D(x, y) + c_\mu \text{ for all } x, y \in D - K_{2\mu}.$$

Assumption (B<sub>4</sub>) holds, in particular, if  $K$  consists of a single point  $x^*$  to which (3.3) is asymptotically stable from  $D$ .

By (B<sub>4</sub>) (ii),  $I_D(x_1, y) = I_D(x_2, y)$  if  $x_1, x_2 \in K$ . Let

$$V = I_D(x, \partial D) = \min_{y \in \partial D} I_D(x, y), \quad x \in K,$$

$$\Sigma = \{y \in \partial D: I_D(x, y) = V, \quad x \in K\}.$$

Theorem 6.1. Let (A<sub>1</sub>), (B<sub>3</sub>), (B<sub>4</sub>) hold. Then

$\text{dist}(\xi^\varepsilon(\tau_D^\varepsilon), \Sigma) \rightarrow 0$  in probability as  $\varepsilon \rightarrow 0$ .

Proof. Given  $\mu > 0$  let  $S$  be open with  $C^2$  boundary  $\partial S$  and  $K \subset S \subset K_\mu$ . Let  $\Delta = D - \bar{S}$ ; and let  $\Lambda = \partial K_{2\mu}$  (we take  $\mu$  small enough that  $\bar{K}_{2\mu} \subset D$ .) Let  $N \subset \partial D$  be closed with  $\Sigma \subset N^0$  and  $\bar{N}^0 = N$  ( $N^0$  = interior of  $N$  relative to  $\partial D$ ). Let  $N^C = \partial D - N$ . There exists  $\rho > 0$  such that, for  $x \in K$ ,

$$I_D(x, N) = V, \quad I_D(x, N^C) > V + 2\rho.$$

By  $(B_4)$  we may choose  $\mu$  small enough that

$$(6.1) \quad \max_{z \in \Lambda} I_\Delta(z, N) < V + \rho < \min_{z \in \Lambda} I_\Delta(z, N^C).$$

Now  $(B_3)$ ,  $(B_4)$  imply  $(B_2)$ . By Lemma 3.1  $(B_2)$  implies  $(B_1)$ . By (6.1) and Theorem 5.1

$$(6.2) \quad \lim_{\varepsilon \rightarrow 0} \frac{q_\Delta^\varepsilon(z, N^C)}{q_\Delta^\varepsilon(z, N)} = 0$$

uniformly for  $z \in \Lambda$ .

Given  $x = \xi^\varepsilon(0)$  in  $D$ , we define random times  $\tau_n, s_n$  as follows

$$\begin{aligned} \tau_0 &= 1^{\text{st}} \text{ time } t \text{ such that } \xi^\varepsilon(t) \in \partial \Delta \\ s_n &= 1^{\text{st}} \text{ time } t > \tau_{n-1} \text{ such that } \xi^\varepsilon(t) \in \Lambda \quad (n \geq 1) \\ \tau_n &= 1^{\text{st}} \text{ time } t > s_n \text{ such that } \xi^\varepsilon(t) \in \partial \Delta \quad (n \geq 1). \end{aligned}$$

Consider the events

$$A_v = \{\tau_D^\varepsilon = \tau_v, \xi^\varepsilon(\tau_D^\varepsilon) \in N\}$$

$$B_v = \{\tau_D^\varepsilon = \tau_v, \xi^\varepsilon(\tau_D^\varepsilon) \in N^c\}.$$

By the strong Markov property

$$P_x(A_v) = E_x[\chi_{\tau^\varepsilon > s_v} q_\Delta^\varepsilon(\xi^\varepsilon(s_v), N)]$$

$$P_x(B_v) = E_x[\chi_{\tau^\varepsilon > s_v} q_\Delta^\varepsilon(\xi^\varepsilon(s_v), N^c)],$$

with  $\chi_G$  the indicator function of an event  $G$ . By (6.2) given  $d > 0$  there exists  $\varepsilon_d$  such that

$$q_\Delta^\varepsilon(z, N^c) \leq d q_\Delta^\varepsilon(z, N)$$

for all  $z \in \Lambda$ ,  $0 < \varepsilon < \varepsilon_d$ . Since  $\xi^\varepsilon(s_v) \in \Lambda$ , we have  $P_x(B_v) \leq d P_x(A_v)$ . Moreover,

$$\sum_v P_x(A_v) \leq \sum_v P_x(A_v \cup B_v) = P_x(\tau_D^\varepsilon < \infty) = 1.$$

Therefore, for  $0 < \varepsilon < \varepsilon_d$ ,

$$P(\xi^\varepsilon(\tau_D^\varepsilon) \in N^c) = \sum_v P_x(B_v) \leq d.$$

Since  $d$  is arbitrary, this proves Theorem 6.1.

Note. Theorem 6.1 is slightly less general than [7, II, Theorem 7.2, p. 361]. We have included it to illustrate the use

of Theorem 5.1. Theorem 6.1 has immediate applications to solutions to the Dirichlet problem  $\mathcal{L}^\varepsilon u^\varepsilon = 0$  in  $D$  with  $u^\varepsilon = U$  on  $\partial D$  [7, II, p. 372]. For instance, if  $\Sigma$  consists of a single point  $y^*$ , then  $u^\varepsilon(x) \rightarrow U(y^*)$  as  $\varepsilon \rightarrow 0$ , for all  $x \in D$ . If  $K$  consists of a single point  $x^* \in D$ , then results equivalent to Theorem 5.1 were used in [17, Lemma 1] to show that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log E_{x^*} \tau_D^\varepsilon = V = \min_{y \in \partial D} I_D(x^*, y).$$

7. Finite time results. Let us now consider the nonautonomous form of (1.1), on a finite time interval  $s \leq t \leq T$ :

$$(7.1) \quad d\xi^\varepsilon = b[t, \xi^\varepsilon(t)]dt + \sqrt{\varepsilon} \sigma[t, \xi^\varepsilon(t)]dw$$

with initial data  $\xi^\varepsilon(s) = x$ . We assume that:

$$(A_2) \quad b(s, x), \sigma(s, x), \text{ and the inverse } \sigma^{-1}(s, x) \\ \text{are bounded, Lipschitz functions on } \mathbb{R}^{n+1}.$$

We fix  $T$  and consider initial data  $(s, x)$  in the cylinder  $Q = (-\infty, T) \times D$ . Given  $N \subset \partial D$  let

$$(7.2) \quad I_Q(s, x, N) = \inf_{\phi, \theta} \int_s^\theta L[t, \phi(t), \dot{\phi}(t)] dt,$$

where the infimum is taken among all  $\phi \in \mathcal{W}^1$  and  $\theta \in [s, T]$  such that  $\phi(s) = x$ ,  $\phi(t) \in \bar{D}$  for  $s \leq t \leq \theta$ ,  $\phi(\theta) \in N$ . The function  $L$  is as in (1.3). Let

$$(7.3) \quad I_Q^\varepsilon(s, x, N) = -\varepsilon \log P_{sx}(\tau_D^\varepsilon \leq T, \xi^\varepsilon(\tau_D^\varepsilon) \in N).$$

Theorem 7.1. Assume  $(A_2)$ , and that  $\bar{N} = \overline{N^0}$ . Then for each  $(s, x) \in Q$ ,

$$\lim_{\varepsilon \rightarrow 0} I_Q^\varepsilon(s, x, N) = I_Q(s, x, N).$$

Theorem 7.1 can be proved by the same method as for Theorem 5.1. The elliptic equation (2.1) is now to be replaced by the backward equation  $0 = g_s^\varepsilon + \mathcal{L}^\varepsilon g^\varepsilon$ , where

$$(7.4) \quad g^\varepsilon(s, x) = E_{sx} \exp \left[ - \frac{\phi[\tau_Q^\varepsilon, \xi^\varepsilon(\tau_Q^\varepsilon)]}{\varepsilon} \right]$$

where  $\tau_Q^\varepsilon = \min(\tau_D^\varepsilon, T)$ ,  $(s, x) \in Q$ . The function  $J^\varepsilon = -\varepsilon \log g^\varepsilon$  satisfies the corresponding dynamic programming equation, in time dependent form. For the finite time problem, assumptions like  $(B_1)$ , or  $(B_3)$  and  $(B_4)$ , are unnecessary. However, without  $(B_3)$  one may have  $I_Q(s, x, N) = 0$ , in which case Theorem 7.1 is uninteresting.

For the special case  $N = \partial D$  we have (see [7, p. 347]):

$$\text{Corollary.} \quad -\lim_{\varepsilon \rightarrow 0} \varepsilon P_{sx}(\tau_D^\varepsilon \leq T) = I_Q(s, x; \partial D).$$

A proof of the Corollary using stochastic control ideas was given in [5].

Instead of repeating for the finite time case details of the argument in §'s 2-5, let us outline a somewhat different proof of Theorem 7.1. A slight refinement of the method, which we

shall not give, shows that the convergence in Theorem 7.1 is uniform on compact subsets of  $Q \cup \{(-\infty, T) \times N^0\}$ .

First of all, let us derive the finite time analogue of Theorem 2.1 in a different way using the Girsanov formula and Itô stochastic differential rule. Let  $v(t)$  be a bounded, non-anticipative control process on  $[s, T]$ , and  $\eta$  the solution of

$$(7.5) \quad d\eta = v(t)dt + \sqrt{\varepsilon} \sigma[t, \eta(t)]dw.$$

Equation (7.5) is obtained from (7.1) by replacing the drift coefficient  $b$  by  $v$ . Let

$$h(t) = \sigma^{-1}(t, \eta(t)) [b(t, \eta(t)) - v(t)].$$

By Girsanov's formula [7, I, Chap. 7]

$$E \exp \left[ - \frac{\Phi(\tau_Q^\varepsilon, \xi^\varepsilon(\tau_Q^\varepsilon))}{\varepsilon} \right] = E \exp \left[ - \frac{1}{\varepsilon} \left\{ \Phi(\theta, \eta(\theta)) + \frac{1}{2} \int_s^T |h(t)|^2 dt \right\} + \frac{1}{\sqrt{\varepsilon}} \int_s^T h(t) ' dv \right],$$

where  $\theta$  is the exit time of  $(t, \eta(t))$  from  $Q$  and  $\tau_Q^\varepsilon$  the exit time of  $(t, \xi^\varepsilon(t))$  from  $Q$ .

Let us require that  $v(t) = b(t, \eta(t))$  for  $t \geq \theta$ . Then

$$\frac{1}{2} |h(t)|^2 = L(t, \eta(t), v(t)), \quad s \leq t < \theta,$$

and  $h(t) = 0$  for  $t \geq \theta$ . By (7.4)

$$g^\varepsilon(s, x) = E \exp(-X),$$

$$(7.6) \quad X = \frac{1}{\varepsilon} \left\{ \Phi(\theta, \eta(\theta)) + \int_s^\theta L(t, \eta(t), v(t)) dt - \sqrt{\varepsilon} \int_s^\theta h(t)' dw \right\}$$

By Jensen's inequality,  $E \exp(-X) \geq \exp[-E(X)]$ , with equality if and only if  $X = \text{constant}$  almost surely. However,  $E(X) = \varepsilon^{-1} \mathcal{J}^\varepsilon(s, x, v)$  where  $\mathcal{J}^\varepsilon$  is defined as in (2.9). Thus,  $g^\varepsilon \geq \exp[\varepsilon^{-1} \mathcal{J}^\varepsilon]$ . To get equality, we define  $v^\varepsilon, \eta^\varepsilon$  as in §2 using the feedback control law (cf. (2.14))

$$V^\varepsilon(s, x) = b(s, x) - a(s, x) (J_x^\varepsilon)'$$

Let  $X^\varepsilon$  be defined by (7.6) when  $\eta = \eta^\varepsilon, v = v^\varepsilon, \theta = \theta^\varepsilon$ , the exit time of  $(t, \eta^\varepsilon(t))$  from  $Q$ . We apply the Itô differential rule to  $J^\varepsilon(t, \eta^\varepsilon(t))$  and use the fact that  $J^\varepsilon = \Phi$  on  $\partial Q$  to conclude that  $X^\varepsilon = \varepsilon^{-1} J^\varepsilon(s, x)$  almost surely. Thus,  $v^\varepsilon(t)$  is an optimal control process. This proves the finite time analogue of Theorem 2.1.

In the last step of this argument we have used the fact that  $J^\varepsilon(s, x)$  is of class  $C^{1,2}(Q) \cap C^1(\bar{Q})$ . This follows from the fact that  $g^\varepsilon = \exp(-\varepsilon^{-1} J^\varepsilon)$  is a positive solution of the linear parabolic equation  $0 = g_s^\varepsilon + \mathcal{L}^\varepsilon g^\varepsilon$ . By [12, Chap. IV, §9],  $g^\varepsilon \in C^{1,2}(Q) \cap C^1(\bar{Q})$  provided the restrictions of  $\Psi(s, x)$  to  $(-\infty, T] \times \partial D$  and to  $\{T\} \times \bar{D}$  are  $C^2$ .

It would be interesting to avoid results from the theory of parabolic partial differential equations altogether in this argument. This could perhaps be done using methods from [2, §6] or [1, Chap. IV].

Let us, for brevity, set  $I_Q^\varepsilon(s, x; N) = I^\varepsilon(s, x)$ ,  $I_Q(s, x; N) = I(s, x)$ . In order to prove Theorem 7.1 it suffices to show that

$$(7.7) \quad \limsup_{\varepsilon \rightarrow 0} I^\varepsilon(s, x) \leq I(s, x)$$

$$(7.8) \quad \liminf_{\varepsilon \rightarrow 0} I^\varepsilon(s, x) \geq I(s, x).$$

Let us first outline a proof of (7.8), which does not involve the Prokhorov compactness criterion and the Skorokhod theorem used in §4. By introducing penalty functions as in §5, it suffices to prove the following Lemma 7.1, which replaces Lemma 4.2. Let

$$\mathcal{T} = ((-\infty, T) \times \partial D) \cup (\{T\} \times \mathbb{R}^n).$$

For any  $s \leq T$  and  $x \in \mathbb{R}^n$ , let

$$(7.9) \quad J(s, x) = \min_{\phi, \theta} \left\{ \int_s^\theta L[t, \phi(t), \dot{\phi}(t)] dt + \Phi[\theta, \phi(\theta)] \right\},$$

where the minimum is taken among all  $\phi \in \mathcal{V}^1$  such that  $\phi(s) = x$  and  $\theta$  such that  $(\theta, \phi(\theta)) \in \mathcal{T}$ . Let  $J^\varepsilon = -\varepsilon \log g^\varepsilon$ .

Lemma 7.1. Assume  $(A_2)$  and that  $\Phi$  is Lipschitz. Then  
 $\liminf_{\varepsilon \rightarrow 0} J^\varepsilon(s, x) \geq J(s, x).$

Proof. For  $\varepsilon > 0$  let  $v^\varepsilon = b - a(J_x^\varepsilon)'$  be the optimal feedback control, as in (2.14). Given  $(s, x) \in Q$ , a brownian motion  $w$ , and  $\{\mathcal{F}_t\}$  to which  $w$  is adapted, we define  $v^\varepsilon(t), \eta^\varepsilon(t), \phi^\varepsilon(t)$

as in §4 by

$$\begin{aligned}\eta^\varepsilon(t) &= \phi^\varepsilon(t) + \sqrt{\varepsilon} \int_s^t \sigma[r, \eta^\varepsilon(r)] dw, \\ \phi^\varepsilon(t) &= x + \int_s^t v^\varepsilon(r) dr, \\ v^\varepsilon(t) &= v^\varepsilon[t, \eta^\varepsilon(t)], \quad s \leq t < \theta^\varepsilon,\end{aligned}$$

with  $\theta^\varepsilon$  the exit time from  $Q$  of  $(t, \eta^\varepsilon(t))$ . For  $t \geq \theta^\varepsilon$ ,  $v^\varepsilon(t) = 0$ . Let

$$(7.10) \quad G^\varepsilon = \int_s^{\theta^\varepsilon} L(t, \eta^\varepsilon(t), v^\varepsilon(t)) dt + \phi(\theta^\varepsilon, \eta^\varepsilon(\theta^\varepsilon)).$$

Let us show that for some constant  $M$

$$(7.11) \quad J(s, x) \leq G^\varepsilon + M ||\eta^\varepsilon - \phi^\varepsilon||,$$

where  $|| \cdot ||$  is the sup norm on  $[s, T]$ . By (1.3) and assumption (A2),  $L(t, y, v) \geq c_1 |v|^2 - c_2$  with  $c_1 > 0$ . Hence there is a constant  $C_1$  such that  $J(s, x) \leq G^\varepsilon$  if  $\int_s^{\theta^\varepsilon} |v^\varepsilon(t)|^2 dt > C_1$ .

Suppose that  $\int_s^{\theta^\varepsilon} |v^\varepsilon(t)|^2 dt \leq C_1$ . An elementary dynamic programming argument shows that

$$(7.12) \quad J(s, x) \leq \int_s^\alpha L(t, \phi, \dot{\phi}) dt + J(\alpha, \phi(\alpha))$$

for any  $\phi \in \mathcal{A}^1$  with  $\phi(s) = x$  and any  $\alpha \in [s, T]$ . Moreover, there exists  $C_2$  such that  $x \in R^n$ ,  $y \in \partial D$  imply

$$(7.13) \quad J(s, x) \leq \phi(s, y) + C_2 |y - x|.$$

This is seen by taking in (7.9) the linear function

$$\phi(t) = x + v(t-s), \quad v = |y-x|^{-1}(y-x)$$

and using the fact that  $\phi$  is Lipschitz. If  $s = T$ , (7.13) holds for all  $x, y$ , since  $J = \phi$  when  $s = T$ . By  $(A_2)$  and (1.3) there exists  $C_3$  such that

$$(7.14) \quad |L(t, y, v) - L(t, z, v)| \leq C_3 |y - z| (1 + |v|^2)$$

for all  $y, z, v \in \mathbb{R}^n$ . By (7.12) with  $\alpha = \theta^\epsilon$ ,

$$\begin{aligned} J(s, x) &\leq \int_s^{\theta^\epsilon} L(t, \phi^\epsilon, v^\epsilon) dt + J(\theta^\epsilon, \phi^\epsilon(\theta^\epsilon)) \\ &\leq G^\epsilon + \int_0^{\theta^\epsilon} |L(t, \phi^\epsilon, v^\epsilon) - L(t, \eta^\epsilon, v^\epsilon)| dt + \\ &\quad + J(\theta^\epsilon, \phi^\epsilon(\theta^\epsilon)) - \phi(\theta^\epsilon, \eta^\epsilon(\theta^\epsilon)). \end{aligned}$$

We use (7.13) with  $(s, x)$  replaced by  $(\theta^\epsilon, \phi^\epsilon(\theta^\epsilon))$  and  $y$  by  $\eta^\epsilon(\theta^\epsilon)$ . In (7.14) we replace  $y$  by  $\phi^\epsilon(t)$  and  $z$  by  $\eta^\epsilon(t)$ . Then (7.11) holds with  $M = C_3(T-s+C_1) + C_2$ .

Since  $v^\epsilon(t)$  is optimal,  $J^\epsilon(s, x) = EG^\epsilon$ . By (7.11)

$$J(s, x) \leq J^\epsilon(s, x) + E ||\eta^\epsilon - \phi^\epsilon||.$$

Since  $\sigma$  is bounded,  $E ||\eta^\epsilon - \phi^\epsilon|| \rightarrow 0$  as  $\epsilon \rightarrow 0$ . This proves Lemma 7.1.

From Lemma 7.1 inequality (7.8) follows, as already indicated.

It remains to consider inequality (7.7). As mentioned in §5, (7.7) is an easy consequence of the first Ventcel-Freidlin estimate. Let us outline a somewhat different argument to get (7.7). Let  $N^* = (-\infty, T) \times N^0$ . We begin with the following:

Lemma 7.2. Given a compact set  $\Lambda \subset Q \cup N^*$  there exist  $\varepsilon_0, C$  such that  $P_{sx}(\tau_D^\varepsilon < T, \xi^\varepsilon(\tau_D^\varepsilon) \in N^0) \geq \exp(-C\varepsilon^{-1})$  for all  $(s, x) \in \Lambda$ ,  $0 < \varepsilon < \varepsilon_0$ .

Lemma 7.2 can be deduced from the Girsanov formula, or by a direct proof (which we omit). We then prove (7.7) as follows.

Consider any  $\phi$  of class  $C^1$  and  $\theta < T$  such that  $\phi(s) = x$ ,  $\phi(t) \in D$  for  $s \leq t < \theta$ ,  $\phi(\theta) \in N^0$ ,  $\phi(t) \notin \bar{D}$  for  $t > \theta$ . Let  $\hat{\eta}^\varepsilon$  be the solution of (7.5) with  $\eta^\varepsilon(s) = x$  using the deterministic control  $v(t) = \dot{\phi}(t)$ . Let  $\Lambda^0$  be open relative to  $\bar{Q}$ , with  $(t, \phi(t)) \in \Lambda^0$  for  $s \leq t \leq \theta$  and  $\bar{\Lambda}^0 \subset Q \cup N^*$ . Now  $I^\varepsilon(s, x)$  satisfies in  $Q$  the dynamic programming equation

$$0 = I_s^\varepsilon + \frac{\varepsilon}{2} \sum_{j,k=1}^n a_{jk} I_{x_j x_k}^\varepsilon + \min_v [L(s, x, v) + I_x^\varepsilon \cdot v].$$

Moreover,  $I^\varepsilon \in C^{1,2}(Q) \cap C^1(\bar{\Lambda}^0)$ . Let  $\beta^\varepsilon$  be the exit time of  $(t, \hat{\eta}^\varepsilon(t))$  from  $\Lambda^0$ . As in (2.11)

$$(7.15) \quad I^\varepsilon(s, x) \leq E \left\{ \int_s^{\beta^\varepsilon} L(t, \hat{\eta}^\varepsilon, \dot{\phi}) dt + I^\varepsilon(\beta^\varepsilon, \hat{\eta}^\varepsilon(\beta^\varepsilon)) \right\}.$$

As  $\varepsilon \rightarrow 0$ ,  $\|\hat{\eta}^\varepsilon - \phi\| \rightarrow 0$  with probability 1 and  $\beta^\varepsilon \rightarrow \theta$  in probability. Moreover,  $I^\varepsilon(t, y) = 0$  for  $(t, y) \in N^*$ ; and by

Lemma 7.2,  $I^\varepsilon(t, y) \leq C$  for all  $(t, y) \in \Lambda^0$ . Then

$$EI^\varepsilon(\beta^\varepsilon, \hat{\eta}^\varepsilon(\beta^\varepsilon)) \leq CP[(\beta^\varepsilon, \hat{\eta}^\varepsilon(\beta^\varepsilon)) \notin N^*],$$

and the right side tends to 0 as  $\varepsilon \rightarrow 0$ . We then have by (7.15)

$$(7.16) \quad \limsup_{\varepsilon \rightarrow 0} I^\varepsilon(s, x) \leq \int_s^\theta L(t, \phi, \dot{\phi}) dt.$$

Since the infimum of the right side among such  $\phi, \theta$  is  $I(s, x)$ , we get (7.7).

Note. The same reasoning can be used to prove (5.5). Given  $T > 0$ , let  $Q = (-\infty, T) \times \Delta$ . Since  $I_\Delta^\varepsilon(x, N) \leq I_Q^\varepsilon(0, x, N)$ , (7.16) with  $D$  replaced by  $\Delta$  implies

$$\limsup_{\varepsilon \rightarrow 0} I_\Delta^\varepsilon(x, N) \leq \int_0^\theta L(\phi, \dot{\phi}) dt$$

for any  $\phi$  such that  $\phi(0) = x$ ,  $\phi(t) \in \Delta$  for  $0 \leq t < \theta$ ,  $\phi(\theta) \in N^0$  (take any  $T > \theta$ ). The infimum of the right side is  $I_\Delta(x, N^0)$ .

A stronger result than Theorem 7.1 (or Theorem 5.1) would be an asymptotic expansion in powers of  $\varepsilon$  of the form

$$(7.17) \quad I_Q^\varepsilon = I_Q + \varepsilon E_1 + \dots + \varepsilon^m E_m + o(\varepsilon^m).$$

Such an expansion can only hold in regions where  $I_Q$  is a smooth solution of the Hamilton-Jacobi equation, constructed from minimizing curves for (7.2). Such regions were called in [4]

regions of strong regularity. It seems likely that the method of [4] can be adapted to obtain (7.17) in regions of strong regularity. However, this matter is not considered here.

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20. Abstract

probability of exit by a given time  $T$ . A new method to obtain such estimates is given, using ideas from stochastic control theory.

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